§ Bonnet-Myers & Cartan-Hadamard Theorem

Recall that a central question in Riem. Geometry is:

Q: Given a complete (M^n, g), how does the curvature(s) reflect the topology of M^n? (E.g.: Gauss-Bonnet)

- one example is Synge Thm: (M^{2n}, g), cpt, orientable, $K > 0 \Rightarrow \pi_1 M = 0$.
- Today: Bonnet-Myers Thm & Cartan-Hadamard Thm.

Bonnet-Myers Thm:

Let (M^n, g) be a complete Riem. manifold.

Suppose $\exists\, r > 0$ st. $\forall\, p \in M$, $\forall\, v \in T_pM$ where $\|v\| = 1$

$$\boxed{Ric^M_p(v,v) \geq \frac{n-1}{r^2} > 0}$$

Ricci curvature of $S^n(r)$

THEN, M is compact and

$$\sup_{p,q \in M}\{d(p,q)\} = \boxed{diam(M^n, g) \leq \pi r}$$

diam $S^n(r)$

Hence, $\pi_1(M)$ is finite.

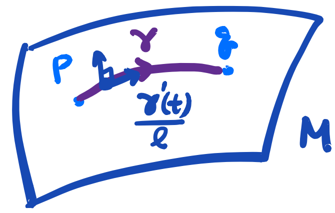
Remark: S.Y. Cheng proved the rigidity case

"Maximal Diameter Thm": $diam\, M = \pi r \Rightarrow (M^n, g) \overset{\text{isometric}}{\cong} (S^n(r), g_{round})$

Proof of Bonnet-Myers:

Idea: Use 2^{nd} variation of geodesics to establish diameter bound.

Take arbitrary points $p, q \in M$. By Hopf-Rinow, $\exists$ minimizing geodesic $\gamma : [0,1] \longrightarrow (M^n, g)$ s.t.



$$\gamma(0) = p \; , \; \gamma(1) = q \; , \quad L(\gamma) = d(p,q) =: \ell$$

Claim: $\ell \leq \pi r$ $\quad (\Rightarrow \text{diam}(M^n, g) \leq \pi r)$

Proof: Argue by contradiction. Suppose NOT, ie $\ell > \pi r$ (*)

γ minimizing $\Rightarrow$ $E''(0) \geq 0 \quad \forall$ variation of γ (#)

Fix a parallel O.N.B. $\left\{ \frac{\gamma'(t)}{\ell}, e_1(t), \dots, e_{n-1}(t) \right\}$ along γ

Define: $V_i(t) := (\sin \pi t)\, e_i(t)$ for $i = 1, \dots, n-1$

Note $V_i(0) = V_i(1) = 0 \Rightarrow$ end-pt fixing variations γ_s^i

2^{nd} variation of energy w.r.t. γ_s^i $\Rightarrow$

$$E_i''(0) = \int_0^1 \langle V_i', V_i' \rangle - \langle R(\gamma', V_i)\gamma', V_i \rangle \, dt$$

$$= - \int_0^1 \langle V_i'' + R(\gamma', V_i)\gamma', V_i \rangle \, dt$$

$$= \int_0^1 \sin^2 \pi t \left[\pi^2 - \ell^2 K_{\gamma(t)}\left(\text{span}\left\{ e_i(t), \tfrac{\gamma'(t)}{\ell} \right\}\right) \right] dt$$

Summing $i = 1, \dots, n-1$,

$$\sum_{i=1}^{n-1} E_i''(0) = \int_0^1 \sin^2 \pi t \left[(n-1)\pi^2 - \ell^2 \overbrace{Ric^M_{\gamma(t)}\left(\frac{\gamma'(t)}{\ell}, \frac{\gamma'(t)}{\ell}\right)}^{\geq \frac{n-1}{r^2} \text{ by assumption}} \right] dt$$

$$\leq \int_0^1 \underbrace{\sin^2 \pi t}_{\geq 0} \underbrace{\left[(n-1)\pi^2 - (n-1)\frac{\ell^2}{r^2} \right]}_{< 0 \text{ by } (*)} dt < 0$$

Thus, $E_i''(0) < 0$ for SOME i, contradicts (#).

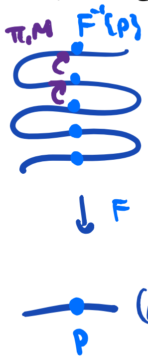
By Hopf-Rinow, $\operatorname{diam}(M, g) \leq \pi r$ + complete $\Rightarrow$ M cpt.

To prove $\pi_1 M$ is finite, consider its "universal cover"

$\pi_1 M \circlearrowright \tilde{M} \leftarrow \pi_1 = 0$

deck transformation

$\tilde{M} \downarrow M$



$(\tilde{M}^n, \overbrace{F^* g}^{\tilde{g}})$ $Ric \geq \frac{n-1}{r^2}$ $\Rightarrow$ $\operatorname{diam}(\tilde{M}, \tilde{g}) \leq \pi r$ & $\tilde{M}$ cpt

$\Downarrow$

(M^n, g) $Ric \geq \frac{n-1}{r^2}$ $\qquad$ $\pi_1 M$ finite.

$\blacksquare$

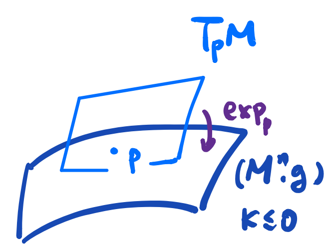
Cartan-Hadamard Thm: Let (M^n, g) be a complete Riem. mfd.

Suppose M has non-positive sectional curvature, i.e.

$$\boxed{K^M \leq 0}$$

THEN, $\exp_p : T_p M \to M$ is a covering map $\forall p \in M$.

Hence, if $\pi_1 M = 0$, then $M^n \overset{\text{diffeo}}{\simeq} T_p M \cong \mathbb{R}^n$.

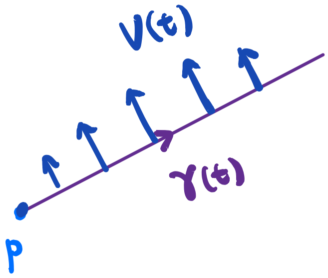


"Sketch of Proof": Idea: Jacobi field estimates.

Step 1: $\nexists$ conjugate pts on ANY geodesic in M

Since (M^n, g) is complete $\xRightarrow[\text{Rinow}]{\text{Hopf-}}$ $\exp_p : T_pM \to M$ is defined.

Let $\gamma : [0,\infty) \to (M^n, g)$ be a geodesic with $\gamma(0) = p$.



Suppose $V(t)$ is a $\overset{\text{normal}}{\vee}$ Jacobi field on γ

st $V(0) = 0$, $V'(0) \neq 0$

Claim: $V(t) \neq 0 \quad \forall t \in (0,\infty)$

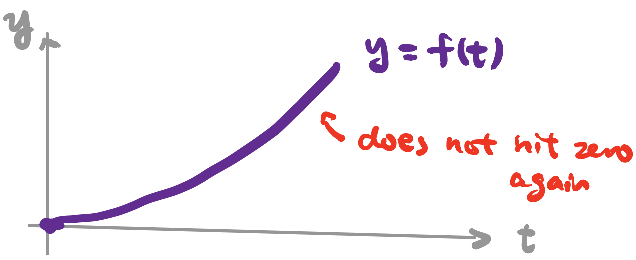
Pf: Consider the function $f(t) := \|V(t)\|^2$

$$f'(t) = 2\langle V(t), V'(t)\rangle$$

$$f''(t) = 2\langle V'(t), V'(t)\rangle + 2\langle V(t), V''(t)\rangle$$

$$= 2\langle V'(t), V'(t)\rangle - 2\underbrace{\langle V(t), R(\gamma'(t), V(t))\gamma'(t)\rangle}_{\leq 0 \text{ by } K \leq 0.}$$

Hence, $f''(t) \geq 0 \quad \forall t \in [0,\infty)$.



$f(0) = \|V(0)\|^2 = 0$

$f'(0) = 2\langle V(0), V'(0)\rangle = 0$

$f''(0) = 2\|V'(0)\|^2 > 0$

Step 2: $\exp_p : T_pM \to M$ is a covering map (w.r.t. some metric)

By Step 1, $d(\exp_p)_v$ is non-singular $\forall v \in T_pM$

$\xRightarrow[\text{IFT}]{}$ $\exp_p : T_pM \to M$ is a local diffeo.

$\Rightarrow$ $\exp_p : (T_pM, \exp_p^* g) \to (M, g)$ local isometry

hence is a covering map.

So, if M is simply-connected, since $\pi_1 T_pM = 0$, then $\exp_p$ must be a diffeomorphism.

$\square$

Since P_t, $\tilde{P}_t$ are isometries, we have $\|\tilde{V}(\ell)\| = \|V(\ell)\|$

Finally, one checks (Ex.) that $\tilde{V}(\ell) = df_q(v)$, i.e. df_q is isometry.

$\square$